Exercise 2.6.2

(No periodic solutions to $\dot{x} = f(x)$) Here's an analytic proof that periodic solutions are impossible for a vector field on a line. Suppose on the contrary that $x(t)$ is a nontrivial periodic solution, i.e., $x(t) = x(t+T)$ for some $T > 0$, and $x(t) \neq x(t+s)$ for all $0 < s < T$. Derive a contradiction by considering $\int_t^{t+T} f(x)\frac{dx}{dt}\,dt$.

Solution

The aim is to show that there are no periodic solutions to

$$\dot{x} = f(x).$$

Suppose that there is a periodic solution to this equation: $x = x(t) \neq \text{constant}$, where

$$x(t) = x(t+T), \quad T > 0$$

for all t, and T is the smallest time for which $x(t)$ repeats itself. Consider

$$\begin{aligned} I &= \int_{t_0}^{t_0+T} f(x)\frac{dx}{dt}\,dt \\ &= \int_{t_0}^{t_0+T} \dot{x}\frac{dx}{dt}\,dt \\ &= \int_{t_0}^{t_0+T} \frac{dx}{dt}\frac{dx}{dt}\,dt \\ &= \int_{t_0}^{t_0+T} \left(\frac{dx}{dt}\right)^2 dt, \end{aligned}$$

which is a positive number since the integrand is nonnegative and $x(t) \neq \text{constant}$. Find I again by integrating by parts.

$$\begin{aligned} I &= \int_{t_0}^{t_0+T} f[x(t)]\frac{dx}{dt}\,dt \\ &= f[x(t)]x(t)\Big|_{t_0}^{t_0+T} - \int_{t_0}^{t_0+T} \frac{d}{dt}\{f[x(t)]\}x(t)\,dt \\ &= f[x(t_0+T)]x(t_0+T) - f[x(t_0)]x(t_0) - \int_{t_0}^{t_0+T} \frac{df}{dx}\frac{dx}{dt}x(t)\,dt \\ &= \cancel{f[x(t_0)]x(t_0)} - \cancel{f[x(t_0)]x(t_0)} - \int_{t_0}^{t_0+T} \frac{df}{dx}\left\{\frac{1}{2}\frac{d}{dt}[x(t)]^2\right\} dt \\ &= -\frac{1}{2}\int_{t_0}^{t_0+T} \frac{df}{dx}\frac{d}{dt}[x(t)]^2\,dt \\ &= -\frac{1}{2}\int_{t_0}^{t_0+T} \frac{d}{dt}f\{[x(t)]^2\}\,dt \end{aligned}$$

Evaluate the integral.

$$\begin{aligned} I &= -\frac{1}{2} f\{[x(t)]^2\}\Big|_{t_0}^{t_0+T} \\ &= -\frac{1}{2} f\{[x(t_0+T)]^2\} + \frac{1}{2} f\{[x(t_0)]^2\} \\ &= -\frac{1}{2} f\{[x(t_0)]^2\} + \frac{1}{2} f\{[x(t_0)]^2\} \\ &= 0 \end{aligned}$$

This is a contradiction because I is supposed to be a positive number. The assumption made initially must then be false. Therefore, there are no periodic solutions to $\dot{x} = f(x)$.